

COMMENTS ON EXCHANGE GRAPHS IN CLUSTER ALGEBRAS

HYUN KYU KIM AND MASAHIRO YAMAZAKI

ABSTRACT. An important problem in the theory of cluster algebras is to compute the fundamental group of the exchange graph. A non-trivial closed loop in the exchange graph, for example, generates a non-trivial identity for the classical and quantum dilogarithm functions. An interesting conjecture, partly motivated by dilogarithm functions, is that this fundamental group is generated by closed loops of mutations involving only two of the cluster variables. We present examples and counterexamples for this naive conjecture, and then formulate a better version of the conjecture for acyclic seeds.

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1. INTRODUCTION AND SUMMARY

Cluster algebra is a mathematical framework introduced by Fomin and Zelevinsky [FZ02, FZ03, FZ07]. Cluster algebra is defined from a skew-symmetrizable $n \times n$ integer matrix B and an n -tuple of coefficients in a fixed semi-field. We start with an initial seed, and we generate other seeds by repeating a combinatorial procedure known as a mutation.

An *exchange graph* Γ for a cluster algebra is a graph whose vertices are labeled by seeds, and whose edges by mutations (an edge connects two vertices if the corresponding two seeds are related by the mutation associated to the edge). The exchange graph is by definition connected.

It is an important problem of theory of the cluster algebras to identify the fundamental group of the exchange graph. For example, we can associate a quantum dilogarithm identity [Rei10, Kel11, KN11] as well as a classical dilogarithm identity [Nak11], to any non-trivial closed loop of the exchange graph. The fundamental group can also be thought of as defining relations for the so-called cluster groupoid.

In this short note, we study the fundamental group of the exchange graph. We first formulate Property (★) in section 2. This property holds for finite-type seeds (section 3),

Date: December 2, 2016.

2010 Mathematics Subject Classification. 13F60.

Key words and phrases. cluster algebra, exchange graph, dilogarithm identity.

whereas there are counterexamples for non-finite-type cases (section 4). In section 5 we formulate our conjecture for acyclic cases (Conjecture 1). We will also discuss implications to quantum dilogarithm functions in section 6.

While many of the results presented here are strictly speaking not new, we here tried to provide a coherent presentation incorporating the known results/examples/counterexamples scattered in the literature, and propose a concrete well-defined conjecture. We hope that our small contribution will facilitate further developments in this exciting subject.

We would like to thank Kyungyong Lee for discussion. MY is supported in part by WPI Research Center Initiative (MEXT, Japan), by JSPS Program for Advancing Strategic International Networks to Accelerate the Circulation of Talented Researchers, by JSPS KAKENHI Grant Number 15K17634, and by JSPS-NRF Joint Research Project.

2. PROPERTY (★)

Let us begin by asking if the following property holds:

Property (★). *The fundamental group for the exchange graph is generated by elements of the form $\mathcal{P}\mathcal{L}\mathcal{P}^{-1}$, where \mathcal{L} is a closed loop (with a base point) obtained by mutating 2 of the n cluster variables while keeping the remaining $n - 2$ variables fixed, \mathcal{P} is an arbitrary path originating at the base point, and \mathcal{P}^{-1} is its inverse.*

Remark 1. *The exchange graph is in general an infinite graph. In the definition of the fundamental group, we allow only finite closed loops.*

Let us explain the motivation for this property, originating from the classical dilogarithm identity.

As already commented before, for a closed loop of the exchange graph we can associate an identity for the classical (Rogers) dilogarithm function $L(x)$.¹ In the notations of [KN11, IY16], this identity takes the form

$$(2) \quad \sum_{i=1}^M \epsilon_i L\left(\frac{y_{k_i}(t)^{\epsilon_i}}{1 + y_{k_i}(t)^{\epsilon_i}}\right) = 0.$$

Here we consider M mutations at (k_1, \dots, k_M) , with the (classical) y -variables after t mutations denoted by $y_i(t)$ (with $i = 1, \dots, n$), while $\epsilon_j = \pm 1$ are the so-called tropical signs. For our purposes it is important that the variables $y_{k_i}(t)$, and hence the arguments $(y_{k_i}(t)^{\epsilon_i})/(1 + y_{k_i}(t)^{\epsilon_i})$ of the classical dilogarithm function, are rational functions of the n cluster variables $\{y_i(0)\}_{i=1}^n$ in the initial seed.

Now, one characteristic identity satisfied by the classical dilogarithm function $L(x)$ is the celebrated five-term identity

$$(3) \quad L(x) + L(y) = L\left(\frac{x(1-y)}{1-xy}\right) + L(xy) + L\left(\frac{y(1-x)}{1-xy}\right),$$

which is known to be the classical dilogarithm identity (2) for the A_2 matrix involving five mutations.

¹ This is defined by

$$(1) \quad L(x) := -\frac{1}{2} \int_0^x dt \left(\frac{\log(1-t)}{t} + \frac{\log t}{1-t} \right).$$

If all the functional identities (including the classical dilogarithm identity of (2)) follows from repeated use of the pentagon (3), that gives a strong indication (albeit not a proof) that Property (\star) holds.² In this connection, it is worth mentioning the result of Wojtkowiak:

Theorem 1 ([Woj96]). *Any functional equation of the dilogarithm with rational functions of one variable as arguments is a consequence of the five-term relation (up to a constant).*

One might be tempted to regard this theorem as a supporting evidence for Property (\star) . Unfortunately, it is known that Theorem 1 does not generalize to the case of multiple variables as arguments (cf. [Zag07]).³ Whether or not Property (\star) holds or not therefore should reflect this subtlety, to say the least.

Remark 2. *The reference [TY14] associates a three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory for a sequence of quiver mutations, for a skew-symmetric matrix B . We can then also associate a non-trivial duality (more precisely an equivalence of the S^3 partition function) between a pair of three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories [TY14, GKRY16]. Property (\star) in this context means that any such duality can be obtained starting with the three-dimensional $\mathcal{N} = 2$ mirror symmetry between supersymmetric quantum electro-dynamics (SQED) and the XYZ model [AHI⁺97], together with gauging of global symmetries and adding superpotential interactions (with monopole operators in general) [DGG13]. In other words, if Property (\star) does not hold then it is an indication that there is a new duality between three-dimensional $\mathcal{N} = 2$ theories unknown in the literature. For this reason Property (\star) is also of great interest to physicists.*

3. FINITE TYPE CASE

Let us discuss the case of finite type seeds, where the exchange graph is a finite graph. Such a seed is classified by Fomin and Zelevinsky [FZ03].

Theorem 2 ([FZ03]). *Property (\star) holds for finite type seeds.*

Proof. The proof below is essentially contained in [FZ03]. However, the significance of their result in the current context of Property (\star) is not emphasized too explicitly in the paper, and hence we find it worth spending a few paragraphs on the proof.

The exchange graph is dual to the cluster complex Δ , which is a $(n - 1)$ -dimensional simplicial complex whose ground set is the set of all cluster variables and whose maximal simplices are the clusters. Namely, a $(d - 1)$ -dimensional simplex is given by a d -element subset of a single cluster. In particular, the top-dimensional simplex, namely an $(n - 1)$ -dimensional simplex, corresponds to a cluster.

The cluster complex Δ coincides with another simplicial complex $\Delta(\Phi)$ defined from the root system Φ for the associated Dynkin type ([FZ03], in particular Theorem 1.13). Now Theorem 1.4 of [CFZ02] (Theorem 3.2 of [FZ03]) says that the (complete) simplicial fan consisting of the cones spanned by simplices of $\Delta(\Phi)$ is the normal fan of a simple n -dimensional convex polytope. Lemma 2.2 of [FZ03] then implies that the fundamental group of the exchange graph is generated by elements of the form $\mathcal{P}\mathcal{L}\mathcal{P}^{-1}$, where \mathcal{L} is a ‘geodesic loop’, \mathcal{P} is an arbitrary path originating at the base point, and \mathcal{P}^{-1} is its inverse.

²We can formulate this problem more intrinsically at the level of the Bloch group. Suppose that we have a set of rational functions $x_i(\vec{t})$ with respect to variables \vec{t} satisfying $\sum_i c_i x_i(\vec{t}) \wedge (1 - x_i(\vec{t})) = 0$. Then $\sum_i c_i [x_i(\vec{t})]$ defines an element of the Bloch group, and the question is if this element is trivial in the Bloch group.

³Five-term identity “almost” determines the function $L(x)$; a one-variable function satisfying the pentagon (3) as well as the inversion relation $L(x) + L(1 - x) = \pi^2/6$ and differentiable three times or more coincides with $L(x)$ [Rog06, section 4]). This result in itself, however, does not guarantee that (2) arises from repeated use of the five-term identity (3).

It therefore remains to identify geodesic loops with mutations involving only two cluster variables. In [FZ03] one picks up a $(n - 2)$ -dimensional simplicial complex D , and define the simplicial complex Δ_D by a quotient. Namely, this is a simplicial complex of the quotient cluster algebras, such that D' is a simplex of Δ_D if and only if $D \cup D'$ is a simplex in Δ . In the language of the cluster variables, this means that we fix $n - 2$ cluster variables, and then consider the mutations on the remaining two variables. The geodesic loop is identified to be a loop inside this 1-dimensional complex Δ_D . This completes the identification. \square

Corollary 1. *For finite type seeds, the loop \mathcal{L} in property (\star) corresponds to a sequence of mutations of rank-two skew-symmetrizable matrices of type either $A_1 \times A_1, A_2, B_2$ or G_2 , and of length 4, 5, 6, 8, respectively.*

Proof. For a finite type seed, we have $|b_{i,j}b_{j,i}| = 0, 1, 2, 3$ for any $i, j = 1, \dots, n$ [FZ03]. The 2×2 matrix defined by $(b_{i,j})$ is then of type $A_1 \times A_1, A_2, B_2$ or G_2 , respectively. Since the matrix is of rank 2, for each case it is then straightforward to repeat mutations to identify the closed loop in the exchange graph, and to see that it has length 4, 5, 6, 8. \square

4. NON-FINITE-TYPE CASES

Let us next move onto non-finite-type cases. For this purpose, it is useful to quote the results for cases where the seed is generated from a signed adjacency matrix of an ideal triangulation of a closed Riemann surface with punctures, as described in [FST08].

Theorem 3 ([FST08],[FT12]). *Property (\star) holds for seeds generated from a signed adjacency matrix of an ideal triangulation of a bordered surface with punctures, except when the surface is a closed surface with exactly two punctures.*

Let us define a graph Γ' to be a graph whose vertex is an ideal triangulation and whose edge connecting two vertices is an operation called a flip, mapping one ideal triangulation to another. The following theorem guarantees that the counterpart of Property (\star) holds for this graph:

Theorem 4 ([FST08], Theorem 3.10⁴). *The fundamental group of Γ' is generated by cycles of length 4 corresponding to pairs of commuting flips, and the cycles of length 5 whose removal would create a pentagonal face.*

There is one cautionary remark, however. As emphasized in [FST08] the graph Γ' is coming from ideal triangulations in general only a subgraph of the exchange graph Γ , and the fundamental group of the former is in general a subgroup of that of the latter. This is because we cannot flip at an edge surrounded by a self-folded triangle, whereas in the cluster algebras we can mutate at any vertex of the quiver diagram.

For this reason, Theorem 4 does not imply Property (\star) . In fact, we can find an example of a surface-type seed which does *not* satisfy Property (\star) :

Proposition 1 ([FST08], Remark 9.19; [FT12]). *Property (\star) does not hold for a seed associated to an ideal triangulation of a closed surface (of genus 1 or higher) with two punctures.*

⁴This theorem is known long before [FST08], see e.g. [Har86] and references in [FST08] for more detailed literature list.

Example 1. *Let us give an example for the genus 1 surface with two punctures. In this case, we can start with the ideal triangulation of Fig. 1, whose associated exchange matrix is given by*

$$(4) \quad B = \begin{pmatrix} 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 \\ 1 & -1 & 0 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 1 \\ -1 & 1 & 1 & 0 & -1 & 0 \end{pmatrix}.$$

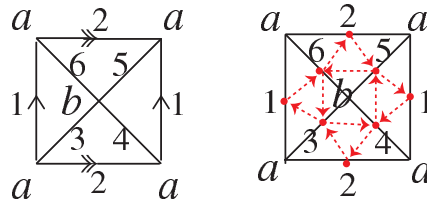


FIGURE 1. ((Left) An ideal triangulation of the torus with two punctures. The left and right (top and bottom) edges are to be identified. The edges are labeled by $1, 2, \dots, 6$, and punctures by a, b . (Right) an associated quiver, whose signed adjacency matrix gives the exchange matrix B in (4).

We found a mutation sequence corresponding to a closed loop in the exchange graph, which is of length 32:

$$(5) \quad 5, 6, 4, 3, 6, 5, 1, 2, 4, 3, 6, 5, 3, 4, 2, 1, 6, 5, 3, 4, 5, 6, 1, 2, 3, 4, 5, 6, 4, 3, 2, 1,$$

For completeness we also have verified numerically that the cluster comes back to itself after this mutation sequence.

To better understand this mutation sequence geometrically, it is useful to use the concept of a tagged triangulation. This concept is introduced in order to allow for a mutation at an edge encircled by a self-folded triangle. In practice, a tagged triangulation is defined to be a maximal collection of pairwise compatible tagged arcs, see [FST08, section 7] for details. In particular, each of the two endpoints of a tagged arc has an extra label, either ‘plain’ or ‘notched’. Graphically we can represent the ‘notched’ labeling by the symbol \bowtie .

The length 32 mutation sequence then can be shown as in Figure 2. Notice that the signatures of tags⁵ around the two punctures are different for all the eight tagged triangulations shown in the figure. This corresponds to the fact that we are going around eight different stratum in the language of [FST08].

Example 2. *We can generalize Example 1 to a general $g > 1$ surface with two punctures.*

Let us triangulate the surface as in Fig. 3. We find the mutation sequence in this case is as follows. First, we consider flip at $4g - 2$ edges, while still in the same stratum:

$$(6) \quad 2g + 2, 2g + 3, \dots, 4g - 1, 4g, \quad 4g + 2, 4g + 3, \dots, 6g.$$

⁵We can associate a signature $+1, 0, -1$ to each puncture, and the set of such numbers define strata [FST08, section 9].

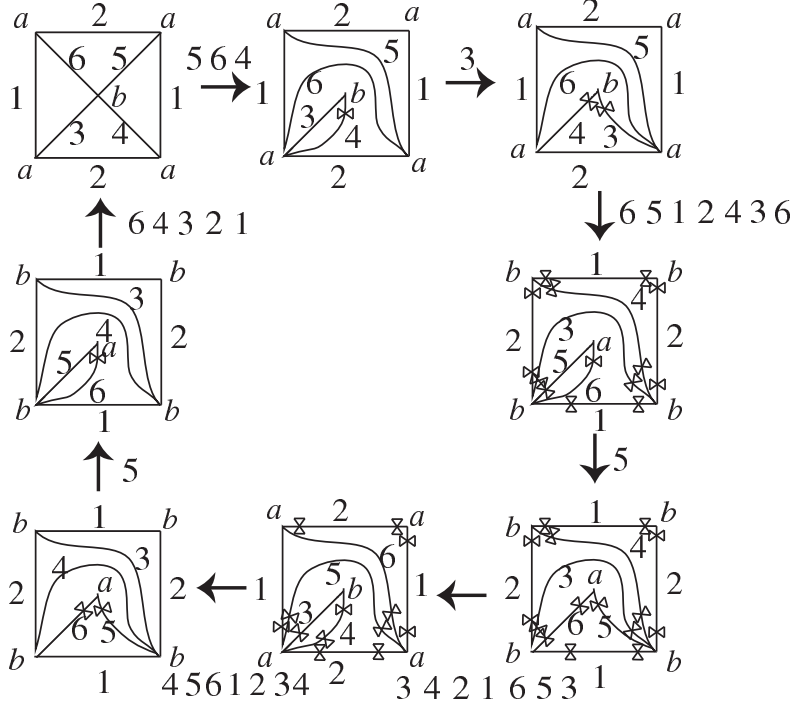


FIGURE 2. A closed loop in the exchange diagram, for a seed corresponding to a tagged triangulation of a genus 1 surface with two punctures. The eight tagged triangulations shown here all have different signatures of tags around the puncture, and belong to different strata [FST08, Remark 9.19].

We then flip at edges $2g+1$ and then $4g+1$, each time changing tags at one of the punctures. Afterwards we flip at edges

$$(7) \quad 6g, 6g-1, \dots, 4g+3, 4g+2, \quad 4g, 4g-1, \dots, 2g+2, \quad 1, 2, 3, 4, \dots, 2g.$$

After all these $10g-2$ flips, the triangulation comes back to itself modulo labels of edges and the punctures interchanged. We can then repeat the similar mutation sequence three more times, and after $4(10g-2) = 40g-8$ steps we are back to the initial tagged triangulation.

5. CONJECTURE

For those seeds not coming from ideal triangulations of closed Riemann surfaces with punctures, there seems to be no general known results applicable in general.

Let us here state our conjecture:

Conjecture 1. *Property (\star) holds for seeds of acyclic type.*

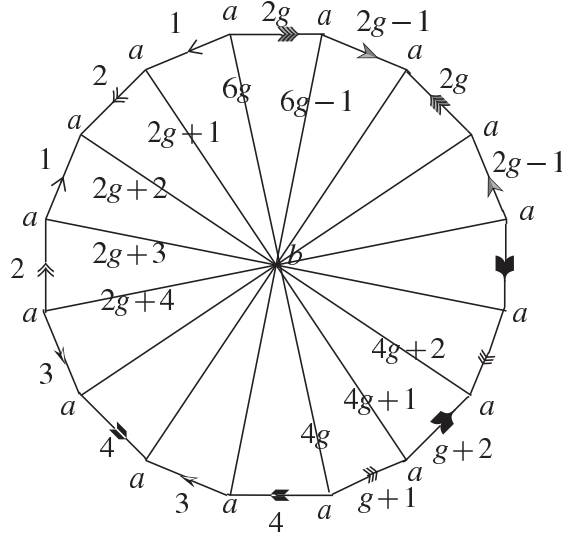


FIGURE 3. An ideal triangulation of a genus g surface with two punctures, shown here is the case of $g = 4$. On the boundary of this $4g$ -gon, edges with the same labels are to be identified and glued together. We have $6g$ edges and two punctures a, b .

Remark 3. Towards the completion of this paper it came to our attention that the same conjecture was made in [War14, Conjecture 8.15]. Note [FZ02] (above Example 7.8) contains a similar conjecture, however without the restriction to acyclic case (recall the counterexample of Proposition 1).

An interesting example is provided by the so-called (G, G') quiver, specified by a pair of Dynkin diagrams [Kel13]. This quiver has a known mutation sequence which sends the seed back to itself (this is the statement of the periodicity of the Y -system [Zam91, KN92, GT96]). We can try to see if this loop of the exchange graph arises from the pentagon. For example, for the (A_3, A_3) quiver the loop of the exchange diagram is of length 36. With the help of computer program we have verified that this follows from the repeated use of closed loop corresponding to rank 2 skew-symmetric matrices of type A_2 and $A_1 \times A_1$. We can repeat similar computations for other small values of k and n . It would be interesting to prove this result for general values of k and n .

6. QUANTUM DILOGARITHM IDENTITIES

As alluded to before, a loop in the exchange graph gives rise to a quantum dilogarithm identity. Let us study the implications of these results at the level of the quantum dilogarithm function.

Definition 1 (Quantum Dilogarithm Function [FK94, FV93]). We define the compact quantum dilogarithm function $\Psi_q(x)$ by⁶

$$(8) \quad \Psi_q(x) := \frac{1}{(-qx; q^2)_\infty}, \quad (x; q)_\infty := \prod_{k=0}^{\infty} (1 - q^k x).$$

⁶ In some literature q^2 is denoted by q . In our notation we always have integer powers of q .

and the non-compact dilogarithm $\Phi^{\hbar}(z)$ by an integral expression valid in the region $|\operatorname{Im}(z)| < \pi(1 + \hbar)$:

$$(9) \quad \Phi^{\hbar}(z) := \exp\left(-\frac{1}{4} \int_{\Omega} \frac{e^{-ipz}}{\sinh(\pi p) \sinh(\pi \hbar p)} \frac{dp}{p}\right),$$

where \hbar is a positive real number and the contour Ω is along the real axis, avoiding the pole at the origin from above along a small half-circle.

The discussion of quantum dilogarithm identity is parallel between compact and non-compact dilogarithms [KN11], and we therefore mostly state only the compact cases, to avoid repetition.

Theorem 5 (Quantum Dilogarithm Identity in Tropical Form [Rei10, Kel11]). *Consider a closed loop of the exchange graph, and let us denote the associated sequence of mutations by the labels k_1, k_2, \dots, k_M . Let c_i ($i = 1, \dots, M$) be the c -vector of the classical y -variables, and let us denote their tropical signs by ϵ_i ($i = 1, \dots, M$). We then have*

$$(10) \quad \Psi_q(Y_{\epsilon_1 c_1})^{\epsilon_1} \cdots \Psi_q(Y_{\epsilon_M c_M})^{\epsilon_M} = 1,$$

where the quantum y -variables Y_{α} with $\alpha = \sum_{i=1}^n \alpha_i e_i \in \mathbb{Z}^n$ satisfy the relation (with $Y_i := Y_{e_i}$)

$$(11) \quad Y_{\alpha+\beta} = q^{\langle \alpha, \beta \rangle} Y_{\alpha} Y_{\beta}, \quad \langle \alpha, \beta \rangle = -\langle \beta, \alpha \rangle := \alpha^T B \beta.$$

These general quantum dilogarithm identities have applications to a wide-ranging topics in mathematics and physics, e.g. references in [KN11, IY16].

Proposition 2. *The dilogarithm identity for rank 2 finite-type skew-symmetrizable matrix B are given as follows:*

For $A_1 \times A_1$ with $Y_1 Y_2 = Y_2 Y_1$,

$$(12) \quad \Psi_q(1) \Psi_q(2) = \Psi_q(1) \Psi_q(1).$$

For A_2 with $Y_1 Y_2 = q^2 Y_2 Y_1$:

$$(13) \quad \Psi_q(1) \Psi_q(2) = \Psi_q(2) \Psi_q(12) \Psi_q(1).$$

For B_2 with $Y_1 Y_2 = q^4 Y_2 Y_1$:

$$(14) \quad \Psi_q(1) \Psi_{q^2}(2) = \Psi_{q^2}(2) \Psi_q(12) \Psi_{q^2}(1^2 2) \Psi_q(1).$$

For G_2 with $Y_1 Y_2 = q^6 Y_2 Y_1$:

$$(15) \quad \Psi_{q^3}(1) \Psi_q(2) = \Psi_q(2) \Psi_{q^3}(12^3) \Psi_q(12^2) \Psi_{q^2}(1^2 2^3) \Psi_q(12) \Psi_{q^3}(1).$$

In the expressions above we used the shorthand notation $\Psi_q(1) = \Psi_q(Y_1)$, $\Psi_q(12) = \Psi_q(Y_{e_1+e_2}) = \Psi_q(q^{-1} Y_1 Y_2)$, $\Psi_q(1^2 2) = \Psi_q(Y_{2e_1+e_2})$, etc. For example, the A_2 identity is the pentagon identity

$$(16) \quad \Psi_q(Y_1) \Psi_q(Y_2) = \Psi_q(Y_2) \Psi_q(q^{-1} Y_1 Y_2) \Psi_q(Y_1).$$

Proof. This is by explicit computation, similar to the D_4 case in Appendix A. \square

Proposition 3. *Let B be a skew-symmetrizable matrix such that its seed is of finite type. Then the associated quantum dilogarithm identities are generated by rank 2 identities in Proposition 2.*

Lemma 1 (Folding Formulas for Quantum Dilogarithm). *We have for the compact dilogarithm*

$$\begin{aligned}
 \Psi_q(x) &= \Psi_{q^2}(qx) \Psi_{q^2}(q^{-1}x) \\
 &= \Psi_{q^3}(q^2x) \Psi_{q^3}(x) \Psi_{q^3}(q^{-2}x) \\
 &= \prod_{j=1}^k \Psi_{q^k}(q^{k-2j+1}x) .
 \end{aligned}
 \tag{17}$$

and for the non-compact quantum dilogarithm

$$\begin{aligned}
 \Phi^{\hbar}(z) &= \Phi^{2\hbar}(z + i\pi\hbar) \Phi^{2\hbar}(z - i\pi\hbar) \\
 &= \Phi^{3\hbar}(z + 2i\pi\hbar) \Phi^{3\hbar}(z) \Phi^{3\hbar}(z - 2i\pi\hbar) \\
 &= \prod_{j=1}^k \Phi^{k\hbar}(z + (k - 2j + 1)i\pi\hbar) .
 \end{aligned}
 \tag{18}$$

Proof. These identities can easily be verified from definitions of the quantum dilogarithm functions (recall (8) and (9)). For example, from (9) we find (in the region $|\operatorname{Im}(z)| < \pi(1 + \hbar)$)

$$\begin{aligned}
 \prod_{j=1}^k \Phi^{k\hbar}(z + (k - 2j + 1)\hbar) &= \exp\left(-\frac{1}{4} \int_{\Omega} \frac{\sum_{i=1}^k e^{-ip(z+i\pi(k-2i+1)\hbar)} dp}{\sinh(\pi p) \sinh(k\pi\hbar p)} \frac{dp}{p}\right) \\
 &= \exp\left(-\frac{1}{4} \int_{\Omega} \frac{e^{-ipz} \sin(k\pi p\hbar) / \sin(\pi p\hbar)}{\sinh(\pi p) \sinh(k\pi\hbar p)} \frac{dp}{p}\right) \\
 &= \exp\left(-\frac{1}{4} \int_{\Omega} \frac{e^{-ipz}}{\sinh(\pi p) \sinh(\pi\hbar p)} \frac{dp}{p}\right) \\
 &= \Phi^{\hbar}(z) .
 \end{aligned}
 \tag{19}$$

The identity can then be analytically continued to the whole complex plane. \square

Lemma 2. *The B_2 and G_2 identities follow from the A_2 identity as well as (17).*

Proof. Let us work out the G_2 quantum dilogarithm identity, since the B_2 case is similar (and easier)⁷. The G_2 identity (15) follows from the D_4 identity

$$\begin{aligned}
 \Psi_q(4)\Psi_q(1)\Psi_q(2)\Psi_q(3) &= \Psi_q(1)\Psi_q(2)\Psi_q(3)\Psi_q(1234) \\
 &\times \Psi_q(234)\Psi_q(134)\Psi_q(124)\Psi_q(1234^2)\Psi_q(14)\Psi_q(24)\Psi_q(34)\Psi_q(4)
 \end{aligned}
 \tag{20}$$

by the substitution $1, 2, 3 \rightarrow 1, 4 \rightarrow 2$ (this is a version of the standard folding trick for non-simply-laced Dynkin diagram). The equation (20) in itself can easily be proven by repeated application of the pentagon relation, or from the general quantum dilogarithm identity (10) (see appendix A). \square

Theorem 6. *Let B be skew-symmetrizable matrix such that its cluster is of finite type. Then the associated quantum dilogarithm identities are generated by the following three identities, and their conjugation by some operators: $A_1 \times A_1$ identity (12), A_2 identity (pentagon) (16), and the folding formula (17) with $k = 2, 3$.*

Proof. This follows from Lemma 2 and Proposition 3. \square

⁷ During the preparation of this work we came to aware that the fact that G_2 identity follows from the pentagon identity is known to some experts, including Gen Kuroki [Kur]. We would like to thank him for correspondence.

Remark 4. *Since we can prove (16), (12) and (17) directly from the definition of the quantum dilogarithm, Theorem 6 gives a direct proof of quantum dilogarithm identities for finite type quivers.*

Remark 5. *By the same arguments Theorem 6 can be immediately generalized to several other versions of quantum dilogarithm identities. This includes the quantum dilogarithm identities in the so-called “universal forms” of [KN11], as well as those for the cyclic dilogarithm [IY16].*

Remark 6. *For seeds originating from four-dimensional $\mathcal{N} = 2$ gauge theories (such as supersymmetric Yang-Mills theory with and without matter), the Kontsevich-Soibelman wall crossing formula [KS08] for the BPS spectrum gives rise to an identity involving an infinite number of quantum dilogarithm (e.g. [DGS11]). However, we have excluded those infinite closed loops from our definition of a fundamental group, see Remark 1.*

APPENDIX A. TROPICAL y -VARIABLES FOR D_4

In this appendix let us explicitly compute the dilogarithm identity for D_4 , as an illustration of the procedure to write down the quantum dilogarithm identity (10).

Let us start with the quiver

$$(21) \quad 1, 2, 3 \longleftarrow 4,$$

which gives a skew-symmetric matrix

$$(22) \quad B = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ +1 & +1 & +1 & 0 \end{pmatrix}.$$

Quantum y -variables Y_1, Y_2, Y_3, Y_4 then satisfy

$$(23) \quad Y_a Y_4 = q^2 Y_4 Y_a, \quad Y_a Y_b = Y_b Y_a, \quad (a, b = 1, 2, 3).$$

Let us consider a mutation sequence $(1, 2, 3, 4, 1, 3, 2, 4, 1, 2, 3, 4, 1, 2, 3, 4)$ of length 16. This represents a closed loop in the exchange graph. Indeed, the classical y -variables (with

$q = 1$) transform as

$$\begin{aligned}
 (24) \quad & Y(0) = (Y_1, Y_2, Y_3, Y_4), \\
 & Y(1) = \left(\frac{1}{Y_1}, Y_2, Y_3, (Y_1 + 1)Y_4 \right), \\
 & Y(2) = \left(\frac{1}{Y_1}, \frac{1}{Y_2}, Y_3, (Y_1 + 1)(Y_2 + 1)Y_4 \right), \\
 & Y(3) = \left(\frac{1}{Y_1}, \frac{1}{Y_2}, \frac{1}{Y_3}, (Y_1 + 1)(Y_2 + 1)(Y_3 + 1)Y_4 \right), \\
 & Y(4) = \left(\frac{(Y_1 + 1)(Y_2 + 1)(Y_3 + 1)Y_4 + 1}{Y_1}, \frac{(Y_1 + 1)(Y_2 + 1)(Y_3 + 1)Y_4 + 1}{Y_2}, \frac{(Y_1 + 1)(Y_2 + 1)(Y_3 + 1)Y_4 + 1}{Y_3}, \frac{(Y_1 + 1)(Y_2 + 1)(Y_3 + 1)Y_4 + 1}{Y_4} \right), \\
 & Y(5) = \left(\frac{Y_1}{(Y_1 + 1)(Y_2 + 1)(Y_3 + 1)Y_4 + 1}, \frac{(Y_1 + 1)(Y_2 + 1)(Y_3 + 1)Y_4 + 1}{Y_2}, \frac{(Y_1 + 1)(Y_2 + 1)(Y_3 + 1)Y_4 + 1}{Y_3}, \frac{(Y_2 + 1)(Y_3 + 1)Y_4 + 1}{Y_1} \right), \\
 & Y(6) = \left(\frac{Y_1}{(Y_1 + 1)(Y_2 + 1)(Y_3 + 1)Y_4 + 1}, \frac{Y_2}{(Y_1 + 1)(Y_2 + 1)(Y_3 + 1)Y_4 + 1}, \frac{(Y_1 + 1)(Y_2 + 1)(Y_3 + 1)Y_4 + 1}{Y_3}, \frac{((Y_1 + 1)(Y_3 + 1)Y_4 + 1)((Y_2 + 1)(Y_3 + 1)Y_4 + 1)}{Y_1} \right), \\
 & Y(7) = \left(\frac{Y_1}{(Y_1 + 1)(Y_2 + 1)(Y_3 + 1)Y_4 + 1}, \frac{Y_2}{(Y_1 + 1)(Y_2 + 1)(Y_3 + 1)Y_4 + 1}, \frac{Y_3}{(Y_1 + 1)(Y_2 + 1)(Y_3 + 1)Y_4 + 1}, \frac{((Y_1 + 1)(Y_2 + 1)Y_4 + 1)((Y_1 + 1)(Y_3 + 1)Y_4 + 1)((Y_2 + 1)(Y_3 + 1)Y_4 + 1)}{Y_1 Y_2 Y_3 Y_4} \right), \\
 & Y(8) = \left(\frac{(Y_1 + 1)(Y_2 + 1)(Y_3 + 1)Y_4^2 + (Y_1 + Y_2 + Y_3 + 2)Y_4 + 1}{Y_2 Y_3 Y_4}, \frac{(Y_1 + 1)(Y_2 + 1)(Y_3 + 1)Y_4^2 + (Y_1 + Y_2 + Y_3 + 2)Y_4 + 1}{Y_1 Y_3 Y_4}, \frac{(Y_1 + 1)(Y_2 + 1)(Y_3 + 1)Y_4^2 + (Y_1 + Y_2 + Y_3 + 2)Y_4 + 1}{Y_1 Y_2 Y_4}, \frac{Y_1 Y_2 Y_3 Y_4}{((Y_1 + 1)(Y_2 + 1)Y_4 + 1)((Y_1 + 1)(Y_3 + 1)Y_4 + 1)((Y_2 + 1)(Y_3 + 1)Y_4 + 1)} \right), \\
 & Y(9) = \left(\frac{Y_2 Y_3 Y_4}{(Y_1 + 1)(Y_2 + 1)(Y_3 + 1)Y_4^2 + (Y_1 + Y_2 + Y_3 + 2)Y_4 + 1}, \frac{(Y_1 + 1)(Y_2 + 1)(Y_3 + 1)Y_4^2 + (Y_1 + Y_2 + Y_3 + 2)Y_4 + 1}{Y_1 Y_3 Y_4}, \frac{(Y_1 + 1)(Y_2 + 1)(Y_3 + 1)Y_4^2 + (Y_1 + Y_2 + Y_3 + 2)Y_4 + 1}{Y_1 Y_2 Y_4}, \frac{Y_1((Y_1 + 1)Y_4 + 1)}{((Y_1 + 1)(Y_2 + 1)Y_4 + 1)((Y_1 + 1)(Y_3 + 1)Y_4 + 1)} \right), \\
 & Y(10) = \left(\frac{Y_2 Y_3 Y_4}{(Y_1 + 1)(Y_2 + 1)(Y_3 + 1)Y_4^2 + (Y_1 + Y_2 + Y_3 + 2)Y_4 + 1}, \frac{Y_1 Y_3 Y_4}{(Y_1 + 1)(Y_2 + 1)(Y_3 + 1)Y_4^2 + (Y_1 + Y_2 + Y_3 + 2)Y_4 + 1}, \frac{(Y_1 + 1)(Y_2 + 1)(Y_3 + 1)Y_4^2 + (Y_1 + Y_2 + Y_3 + 2)Y_4 + 1}{Y_1 Y_2 Y_4}, \frac{((Y_1 + 1)Y_4 + 1)((Y_2 + 1)Y_4 + 1)}{Y_3 Y_4((Y_1 + 1)(Y_2 + 1)Y_4 + 1)} \right), \\
 & Y(11) = \left(\frac{Y_2 Y_3 Y_4}{(Y_1 + 1)(Y_2 + 1)(Y_3 + 1)Y_4^2 + (Y_1 + Y_2 + Y_3 + 2)Y_4 + 1}, \frac{Y_1 Y_3 Y_4}{(Y_1 + 1)(Y_2 + 1)(Y_3 + 1)Y_4^2 + (Y_1 + Y_2 + Y_3 + 2)Y_4 + 1}, \frac{Y_1 Y_2 Y_4}{(Y_1 + 1)(Y_2 + 1)(Y_3 + 1)Y_4^2 + (Y_1 + Y_2 + Y_3 + 2)Y_4 + 1}, \frac{((Y_1 + 1)Y_4 + 1)((Y_2 + 1)Y_4 + 1)((Y_3 + 1)Y_4 + 1)}{Y_1 Y_2 Y_3 Y_4^2} \right), \\
 & Y(12) = \left(\frac{Y_4 + 1}{Y_1 Y_4}, \frac{Y_4 + 1}{Y_2 Y_4}, \frac{Y_4 + 1}{Y_3 Y_4}, \frac{Y_1 Y_2 Y_3 Y_4^2}{((Y_1 + 1)Y_4 + 1)((Y_2 + 1)Y_4 + 1)((Y_3 + 1)Y_4 + 1)} \right), \\
 & Y(13) = \left(\frac{Y_1 Y_4}{Y_4 + 1}, \frac{Y_2 Y_4}{Y_4 + 1}, \frac{Y_3 Y_4}{Y_4 + 1}, \frac{Y_2 Y_3 Y_4}{((Y_2 + 1)Y_4 + 1)((Y_3 + 1)Y_4 + 1)} \right), \\
 & Y(14) = \left(\frac{Y_1 Y_4}{Y_4 + 1}, \frac{Y_2 Y_4}{Y_4 + 1}, \frac{Y_3 Y_4}{Y_4 + 1}, \frac{Y_2}{(Y_3 + 1)Y_4 + 1} \right), \\
 & Y(15) = \left(\frac{Y_1 Y_4}{Y_4 + 1}, \frac{Y_2 Y_4}{Y_4 + 1}, \frac{Y_3 Y_4}{Y_4 + 1}, \frac{1}{Y_4} \right), \\
 & Y(16) = (Y_1, Y_2, Y_3, Y_4).
 \end{aligned}$$

The c -vectors $c_i := c(y_{k_i}(t))$ and tropical signs ϵ_i are then computed to be

$$\begin{aligned}
 (25) \quad & c_1 = (1, 0, 0, 0), \quad \epsilon_1 = +, \\
 & c_2 = (0, 1, 0, 0), \quad \epsilon_2 = +, \\
 & c_3 = (0, 0, 1, 0), \quad \epsilon_3 = +, \\
 & c_4 = (0, 0, 0, 1), \quad \epsilon_4 = +, \\
 & c_5 = (1, 0, 0, 0), \quad \epsilon_5 = -, \\
 & c_6 = (0, 1, 0, 0), \quad \epsilon_6 = -, \\
 & c_7 = (0, 0, 1, 0), \quad \epsilon_7 = -, \\
 & c_8 = (1, 1, 1, 1), \quad \epsilon_8 = -, \\
 & c_9 = (0, 1, 1, 1), \quad \epsilon_9 = -, \\
 & c_{10} = (1, 0, 1, 1), \quad \epsilon_{10} = -, \\
 & c_{11} = (1, 1, 0, 1), \quad \epsilon_{11} = -, \\
 & c_{12} = (1, 1, 1, 2), \quad \epsilon_{12} = -, \\
 & c_{13} = (1, 0, 0, 1), \quad \epsilon_{13} = -, \\
 & c_{14} = (0, 1, 0, 1), \quad \epsilon_{14} = -, \\
 & c_{15} = (0, 0, 1, 1), \quad \epsilon_{15} = -, \\
 & c_{16} = (0, 0, 0, 1), \quad \epsilon_{16} = -.
 \end{aligned}$$

By substituting these data into the (10), we obtain the D_4 quantum dilogarithm identity (20).

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(HK) SCHOOL OF MATHEMATICS, KOREA INSTITUTE FOR ADVANCED STUDY, SEOUL 02455, KOREA
E-mail address: `hkim(at)kias.re.kr`

(MY) KAVLI IPMU (WPI), UNIVERSITY OF TOKYO, CHIBA 277-8583, JAPAN; AND CENTER FOR FUNDAMENTAL LAWS
OF NATURE, HARVARD UNIVERSITY, CAMBRIDGE MA 02138, USA
E-mail address: `masahito.yamazaki(at)ipmu.jp`